A Field Theory for Partially Polarized Quantum Hall States

T.H. Hansson[1], A. Karlhede[1], J.M. Leinaas[2] and U. Nilsson[3]

[1] *Department of Physics, Stockholm University, Box 6730, S-11385 Stockholm, Sweden*
[2] *Deptartment of Physics, University of Oslo, P.O. Box 1048 Blindern, N-0316 Oslo, Norway*
[3] *Department of Applied Mathematics, University of Waterloo, Ontario, Canada, N2L 3G1*

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We propose a new effective field theory for partially polarized quantum Hall states. The density and polarization for the mean field ground states are determined by couplings to two Chern-Simons gauge fields. In addition there is a σ-model field, $\hat{m}$, which is necessary both to preserve the Chern-Simons gauge symmetry that determines the correlations in the ground state, and the global SU(2) invariance related to spin rotations. For states with non zero polarization, the low energy dynamics is that of a ferromagnet. In addition to spin waves, the spectrum contains topological solitons, or skyrmions, just as in the fully polarized case. The electric charge of the skyrmions is given by $Q_{el} = \nu P Q_{top}$, where ν is the filling fraction, P the magnitude of the polarization, and Q_{top} the topological charge. For the special case of full polarization, the theory involves a single scalar field and a single Chern-Simons field in addition to the σ-model field, $\hat{m}$. We also give a heuristic derivation of the model lagrangians for both full and partial polarization, and show that in a mean field picture, the field $\hat{m}$ is necessary in order to take into account the Berry phases originating from rotations of the electron spins

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I. INTRODUCTION

It is known that at low electron densities fractional quantum Hall (QH) ground states at certain fractions such as 2/5 and 4/3 are not fully polarized. This has been established by studying the transport properties at a fixed filling fraction as a function of the applied magnetic field $\vec{B}$ (either by tilting the field or by changing the electron density) [1–3]. Plateaux that are destroyed by increasing the magnetic field, *i.e.* by increasing the Zeeman gap, are natural candidates for fractional QH states with non-maximal polarization, and observed crossover behavior is consistent with transitions from unpolarized to partially or fully polarized states. This picture is supported by numerical calculations (exact diagonalization of few electron systems) and theoretical considerations based on Halperin wave-functions and hierarchical schemes [4,5].

In another line of development progress has been made in understanding spin effects in fully polarized systems. Sondhi *et al.* predicted that the lowest energy charged excitations are skyrmions [10]. This was then experimentally confirmed by measuring the excess spin of these quasiparticles [6–9].

An important tool in studying skyrmions (and spin textures in general) is effective theories for the spin degrees of freedom. For the simplest case of a fully polarized state, the effective theory is a non-linear σ-model given by,

$$\mathcal{L}_{eff} = \bar{\rho}\mathcal{L}_{kin} - \frac{\kappa\bar{\rho}}{4}(\vec{\nabla}\hat{m})^2$$
$$- V(\bar{\rho} + \delta\rho) + \frac{1}{2}\mu_e\bar{\rho}\vec{B}\cdot\hat{m} \qquad . \tag{1.1}$$

Here $\vec{B}$ is the magnetic field, $V(\delta\rho)$ is the effective Coulomb potential and $\mathcal{L}_{kin}$ is the kinetic term for a ferromagnet, defined by its variation, $\delta\mathcal{L}_{kin}/\delta\hat{m}^i = \epsilon^{ijk}\hat{m}^j\partial_0\hat{m}^k$.[1] The unit vector $\hat{m}$ describes the magnetization and the deviation $\delta\rho$ from the ground state charge density $\bar{\rho}$ is proportional to the topological (Pontryagin) charge density $q = \hat{m}\cdot(\partial_x\hat{m}\times\partial_y\hat{m})/4\pi$: $\delta\rho = \nu q$, where ν is the filling fraction. In the case of a single filled Landau level, $\nu = 1$, the effective lagrangian (1.1) can be derived from first principles [11–13].

In this paper we present an effective theory that has partially polarized ground states; the excitations corresponding to fluctuations in the density and the magnitude of the polarization both have large gaps; the low-energy excitations are ferromagnetic spin waves with the expected Zeeman gap; the effective low energy lagrangian is again a σ-model, which in addition to spin waves also has skyrmion solutions; in the limit of zero polarization the spin waves decouple, and the electric charge of the skyrmions vanishes. Our model is a natural, and in a sense minimal, extension of the Chern-Simons-Landau-Ginzburg model described in ref. 14. That model had partially polarized ground states, but did not account correctly for the low-energy excitations, whereas the present model does. The limiting case of full polarization is treated separately, and the effective low energy lagrangian is again a σ-model.

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[1] $\mathcal{L}_{kin}$ cannot be written as a local function of $\hat{m}$. It has the same form as the action for a charged particle on a sphere moving in the field of a unit magnetic monopole.

We do not have a firm microscopic justification of our model lagrangian, but we will provide a heuristic derivation which we believe captures important pieces of the physics.

The paper is organized as follows. In the next section we present our model lagrangian and discuss its properties. In section 3 we derive the effective low energy lagrangian which is of sigma model type, and derive the spin wave spectrum. Section 4 deals with the skyrmion solutions, and section 5 with the microscopic derivation of the model lagrangian with a special discussion of the case of full polarization. We conclude in section 6 with some comments about the status of the microscopic derivation, and some general remarks.

II. THE MODEL LAGRANGIAN

Consider a two dimensional electron gas subject to a magnetic field of constant magnitude and direction, $\vec{B} = \nabla \times \vec{A}$. The magnetic field is in general tilted relative to a normal vector of the electron plane. We model this system with the following lagrangian (density),

$$\mathcal{L} = \phi^\dagger i D_0 \phi - \frac{1}{2m_e}|\vec{D}\phi|^2 - \frac{1}{2\pi} l_{\alpha\beta}\epsilon_{\mu\nu\sigma}a_\alpha^\mu \partial^\nu a_\beta^\sigma$$
$$- V(\rho_\uparrow, \rho_\downarrow) - \frac{V_0}{2}(\partial_i \vec{S})^2 + \mu_e \vec{S} \cdot \vec{B} \quad , \qquad (2.1)$$

where the covariant derivatives D_0 and $\vec{D}$ are given by[2]

$$iD^\mu = i\partial^\mu + a_1^\mu + a_2^\mu \,\vec{\sigma} \cdot \hat{m} + A^\mu - \frac{1}{2}(\hat{m} \times \partial^\mu \hat{m}) \cdot \vec{\sigma} \ , \quad (2.2)$$

and the spin density $\vec{S}$ is related to the unit vector field $\hat{m}$ by $\vec{S} = (1/2)\,\hat{m}\,\phi^\dagger\vec{\sigma}\cdot\hat{m}\,\phi$, as we shall discuss below. The densities $\rho_\uparrow$ and $\rho_\downarrow$ are defined with respect to the quantization direction $\hat{m}$: $\rho(\hat{m}\cdot\hat{n}) = \phi^\dagger\hat{m}\cdot\vec{\sigma}\phi = \rho_\uparrow - \rho_\downarrow$, where $\rho = \phi^\dagger\phi = \rho_\uparrow + \rho_\downarrow$ and $\rho\hat{n} = \phi^\dagger\vec{\sigma}\phi$.

The field content differs from that in the conventional Landau-Ginzburg description [15,16] in that in addition to the two-component complex *bosonic* field ϕ, and the Chern-Simons (CS) fields a_α^μ, there is also a vector field $\hat{m}$ describing the direction of the spin polarization. We use a notation where $i, j... = 1, 2$; $\mu, \nu.... = 0, 1, 2$; m_e is the electron mass; μ_e is the effective magnetic moment of the electron, V_0 is an interaction parameter and $\hbar = e = c = 1$. The elements of the symmetric matrix l^{-1} are

integers whose diagonal elements are both either even or odd.[3]

It is convenient to decompose ϕ as $\phi = \varphi\chi$, where $\chi^\dagger\chi = 1$, $\varphi = \sqrt{\rho}$, and the CP(1) field χ is related to the vector $\hat{n}$ by $\hat{n} = \chi^\dagger\vec{\sigma}\chi$. The two gauge potentials a_1^μ and a_2^μ couple to the charge and the $\hat{m}$-component of the density $\rho\hat{n}$ respectively. The degrees of freedom in χ can conveniently be thought of as the two angles describing the direction of $\hat{n}$ plus an additional overall phase, $e^{i\theta(x)}$. (Note that it is $\hat{m}$ and not $\hat{n}$ that is identified with the local direction of polarization.)

We now discuss the symmetry properties of (2.1) by considering the following transformations,

$$\chi \to e^{i\alpha(x)}\chi \ , \quad a_1^\mu \to a_1^\mu + \partial^\mu\alpha(x) \qquad (2.3)$$

$$\chi \to e^{i\beta(x)\vec{\sigma}\cdot\hat{m}(x)}\chi \ , \quad a_2^\mu \to a_2^\mu + \partial^\mu\beta(x) \qquad (2.4)$$

$$\chi \to e^{\frac{i}{2}\vec{k}\cdot\vec{\sigma}}\chi \ , \quad \hat{m} \to e^{i\vec{k}\cdot\vec{L}}\hat{m} \quad , \qquad (2.5)$$

where the 3×3 matrix $\vec{L}$ is the angular momentum in the vector representation. While ρ is invariant under the two U(1) gauge transformations (2.3) and (2.4), the unit vector $\hat{n}$ is only invariant under the one related to a_1, while under the one related to a_2, it rotates around the $\hat{m}$-axis as $\hat{n} \to e^{2i\beta(x)\vec{L}\cdot\hat{m}}\hat{n}$. From this follows that both the densities, $\rho_\uparrow$ and $\rho_\downarrow$, and the polarization, $P = \hat{m}\cdot\hat{n} = \cos\alpha$, are invariant under the two gauge transformations and, consequently, so are also the (in general non-local) potential $V(\rho_\uparrow, \rho_\downarrow)$ and the Zeeman term in (2.1). A simple calculation shows that the covariant derivative (2.2) transforms homogeneously under both gauge transformations, and thus the full lagrangian (2.1) is gauge invariant.

In the limit of vanishing Zeeman coupling, *i.e.* $\mu_e = 0$, the lagrangian is also invariant under the *global* SU(2) symmetry (2.5), corresponding to simultaneous constant rotations of ϕ and $\hat{m}$. Using the SU(2) invariant part of the action alone we can use the Noether procedure to derive the generator $\vec{S}$, which is the integral of the spin density $\vec{S}$,[4]

$$\vec{S} = \int d^2x\, \vec{S} = \frac{1}{2}\int d^2x\, \rho(\hat{m}\cdot\hat{n})\hat{m} \qquad . \qquad (2.6)$$

From this it follows that $\hat{m}$ should be identified with the direction, and $\hat{m}\cdot\hat{n}$ with the magnitude, of the polarization.

A homogeneous ground state of the model (2.1) can be found in the following way. We choose Coulomb gauge $(\vec{\nabla}\cdot\vec{a}_\alpha^i = 0)$ and look for a solution which minimizes each

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[2] The three-vector notation is only for notational convenience; the metric is Euclidian, *i.e.* $D^\mu = D_\mu$, and all signs are written explicitly.

[3] The generalized Halperin (m_1, m_2, n) states correspond to the following values for $l_{\alpha\beta}$: $\Delta l_{11} = m_1 + m_2 - 2n$, $\Delta l_{22} = m_1 + m_2 + 2n$ and $\Delta l_{12} = m_2 - m_1$, where $\Delta = m_1 m_2 - n^2$.

[4] Note that $\vec{S}$ gets contributions from variations both in ϕ and $\hat{m}$.

of the terms of the Hamiltonian separately. The Zeeman energy and the gradient energy $\sim (\nabla \hat{S})^2$ are minimized by taking ϕ as an arbitrary constant, and $\hat{m}$ equal to a constant unit vector $\hat{m}^0$ pointing in the direction of $\vec{B}$. In the basis where $\vec{\sigma} \cdot \hat{m}$ is diagonal, the two first terms in (2.1) are diagonal and describe two scalar fields ϕ_- and ϕ_+ coupled to the statistical vector potentials $\vec{a}_- = \vec{a}_1 - \vec{a}_2$ and $\vec{a}_+ = \vec{a}_1 + \vec{a}_2$ respectively. The kinetic energy is minimized by $\phi = const.$, and $\vec{a}_+ = \vec{a}_- = -\vec{A}$. Varying $\mathcal{L}$ with respect to a_α^0 gives the constraints $\pi\rho = -l_{11}b_1 - l_{12}b_2$ and $\pi\rho(\hat{m}\cdot\hat{n}) = -l_{12}b_1 - l_{22}b_2$. The solution corresponding to minimal kinetic energy therefore is

$$\rho = l_{11}B_\perp/\pi \equiv \bar{\rho}$$
$$P = \hat{n} \cdot \hat{m} = \cos \alpha = l_{12}/l_{11} \equiv \bar{P} , \qquad (2.7)$$

where $B_\perp$ is the component of $\vec{B}$ perpendicular to the plane. In order for the density $\bar{\rho}$ to correspond to minimal potential energy, a chemical potential has to be included in the lagrangian. The value of the chemical potential is then fixed by this requirement. Similarly, the requirement that the polarization $\bar{P}$ should correspond to minimal potential energy will fix the value of the Zeeman term. However, one should note that the ground state (2.7) is supposed to exist for a range of values of the chemical potential and the Zeeman energy. This is because a change in these quantities can be absorbed in a change in a_+^0 and a_-^0. Such a change will increase the energy, but not change the ground state until the gap of the excitation energy is exceeded. As in ref. 14, we interpret this mean field ground state as a quantum Hall state with filling fraction $\nu = 2\pi\bar{\rho}/B_\perp = 2l_{11}$, and polarization $\bar{P}$.

III. THE EFFECTIVE σ-MODEL

Since the mean field ground state given above spontaneously breaks the approximate SU(2) symmetry of the model, we expect Goldstone modes with a gap given by the Zeeman energy $\mu_e B$. These modes are spin waves where both $\hat{m}$ and $\hat{n}$ vary, but the total density, ρ, and the magnitude of the polarization, $\hat{m} \cdot \hat{n}$, remain fixed. To find these modes, we parametrize ϕ as:

$$\phi = \sqrt{\rho}e^{i\vartheta}e^{i\frac{\alpha}{2}\vec{\sigma}\cdot\hat{e}_1(\theta)}\chi_{\hat{m}} \qquad , \qquad (3.1)$$

where the spinor $\chi_{\hat{m}}$ is choosen so that

$$\chi_{\hat{m}}^\dagger \vec{\sigma} \chi_{\hat{m}} = \hat{m} \qquad , \qquad (3.2)$$

and we have introduced an orthonormal basis $(\hat{m}, \hat{e}_1, \hat{e}_2)$. The four degrees of freedom in the complex two-spinor field ϕ are coded in the density ρ, and the three angles α, ϑ and θ. Since α is measured relative to the vector $\hat{m}$ ($\cos \alpha = \hat{m} \cdot \hat{n}$) we expect high frequency modes for

the fluctuations in α as well as in ρ. θ parametrizes rotations of $\hat{n}$ around $\hat{m}$ and can be removed by an a_2 gauge transformation (2.4); similarly, ϑ can be removed by an a_1 gauge transformation (2.3). Next we introduce the constant basis $(\hat{m}^0, \hat{e}_1^0, \hat{e}_2^0)$, and taking $\hat{m}^0 = \hat{z}$ we have the following explicit parametrization,

$$\hat{m} = R(\vec{k})\hat{m}^0 = (\sin k \cos \beta, \sin k \sin \beta, \cos k) \qquad (3.3)$$
$$\hat{e}_1^0 = \hat{k} = (-\sin \beta, \cos \beta, 0) \qquad (3.4)$$

and (with $\hat{e}_2^0 = \hat{m} \times \hat{k}$),

$$\hat{e}_1 = \cos \theta \hat{e}_1^0 + \sin \theta \hat{e}_2^0 \qquad . \qquad (3.5)$$

We also define the topological vector potential

$$\tilde{a}^\mu = \chi_{\hat{m}}^\dagger i\partial^\mu \chi_{\hat{m}} = \sin^2 \frac{k}{2}\partial^\mu \beta \qquad . \qquad (3.6)$$

It is now only a matter of algebra to rewrite (2.1) in the following form,

$$\mathcal{L} = \rho[a_1^0 + \cos \alpha\, a_2^0 + \cos \alpha\, \tilde{a}^0]$$
$$- \frac{\rho}{8m_e}[\frac{1}{\rho}(\vec{\nabla}\rho)^2 + (\vec{\nabla}\alpha)^2] - V(\rho_\uparrow, \rho_\downarrow)$$
$$- \frac{\rho}{2m_e}\{[\vec{a}_1 + \vec{A} + \cos\alpha(\vec{a}_2 + \vec{\tilde{a}})]^2 + \sin^2\alpha(\vec{a}_2 + \vec{\tilde{a}})^2\}$$
$$- \frac{1}{2\pi}l_{\alpha\beta}\epsilon_{\mu\nu\sigma}a_\alpha^\mu\partial^\nu a_\beta^\sigma - \frac{V_0}{2}(\partial_i\vec{S})^2 + \mu_e\vec{B}\cdot\vec{S} , \qquad (3.7)$$

where we fixed a unitary gauge by $\vartheta = 0$ and $\hat{e}_1(\theta) = \hat{k}$ (the fields ϑ and θ were absorbed in the longitudinal parts of a_1 and a_2 respectively), and also made the variable changes $a_1^\mu - \frac{1}{2}\partial^\mu\beta \to a_1^\mu$ and $a_2^\mu + \frac{1}{2}\partial^\mu\beta \to a_2^\mu$.

It is easily verified that the topological vector potential $\tilde{a}^\mu$ is related to the ferromagnetic kinetic term in (1.1), and the Pontryagin density, q, by,

$$\mathcal{L}_{kin} = \tilde{a}^0 \qquad (3.8)$$
$$2\pi q = \tilde{b} = \vec{\nabla} \times \vec{\tilde{a}} \qquad . \qquad (3.9)$$

Note that in the parametrization (3.3), $\mathcal{L}_{kin}$ becomes local at the expense of introducing an arbitrary fixed direction $\hat{m}^0$.

To study the fluctuations around the mean field solution, we decompose the gauge fields into transverse and longitudinal parts,

$$\vec{a}_\alpha = \vec{a}_\alpha^T + \vec{\nabla}\theta_\alpha \qquad , \qquad (3.10)$$

use the constraints to express the transverse components in terms of densities, and expand (3.7) to quadratic order in the small parameters θ_α, $\delta\rho$ and $\delta\alpha$. Note that since the lagrangian is first order in time derivatives, these four variables describe only two independent modes corresponding to the fluctuations in the density and the magnitude of the polarization. As expected, these modes

are precisely those of the model with fixed $\hat{m}$ and have gaps given by,

$$\omega_\rho = \omega_c \tag{3.11}$$

$$\omega_\alpha = 2\pi(l^{-1})_{22}\nu\sin^2\alpha\,\omega_c \quad , \tag{3.12}$$

where $\omega_c = B_\perp/m_e$ is the cyclotron frequency. Note that full polarization ($\alpha = 0$) is a special case, which will be treated separately in section 5 . In the two-component formulation it corresponds to a non-invertible l-matrix and the second mode ω_α is not present.

Since the modes (3.11) and (3.12) have large gaps, the corresponding (fast) variables θ_α, $\delta\rho$ and $\delta\alpha$ can be integrated out to give an effective lagrangian in the low-energy variable $\hat{m}$. As a lowest order approximation this can be done by fixing the fast variables through the requirement of minimal energy in a general background field $\hat{m}(x)$. The corresponding ground state is, for a slowly varying field $\hat{m}(x)$, characterized by vanishing fields $b_1 + B_\perp = 0$ and $b_2 + \tilde{b} = 0$. In a general background this implies that the charge and spin densities are not constant, but are related to the topological density q of the background field in the following way,

$$\rho = \bar{\rho} + \frac{l_{12}}{\pi}\tilde{b} = \bar{\rho} + \nu\cos\bar{\alpha}\,q$$

$$\rho\cos\alpha = \bar{\rho}\cos\bar{\alpha} + \frac{l_{22}}{\pi}\tilde{b} = \bar{\rho}\cos\bar{\alpha} + 2l_{22}q \quad . \tag{3.13}$$

With these expressions inserted in the lagrangian one obtains for the effective lagrangian of the low-energy variable $\hat{m}$,

$$\mathcal{L}_\sigma = \rho\cos\alpha\,\mathcal{L}_{kin} - \frac{V_0}{2}(\rho\cos\alpha)^2(\vec{\nabla}\hat{m})^2$$

$$+ \frac{\mu_e}{2}\cos\alpha\,\vec{B}\cdot\hat{m} - V(\rho_\uparrow,\rho_\downarrow) \quad , \tag{3.14}$$

where ρ and $\rho_\uparrow - \rho_\downarrow = \rho\cos\alpha$ are determined by $q(\hat{m})$ via (3.13). (Derivative terms in q have here been neglected.) Higher order terms in the loop expansion give derivative corrections to this result. From (3.14) we obtain the following spin wave dispersion relation to second order in the momentum p,

$$\omega = \mu_e B + \frac{\kappa}{\cos\bar{\alpha}}p^2 \quad , \tag{3.15}$$

with $\kappa = (V_0/2)\rho\cos^2\bar{\alpha}$. The effective lagrangian (3.14) is valid for full as well as for partial polarization. Note that for $\alpha = 0$, $i.e.$ the fully polarized case, we retain the result of Sondhi $et\ al.$ [10] with correct normalization of the gap. For $\alpha = \pi/2$ the kinetic term in (3.14) vanishes and there is no propagating spin wave.

IV. SKYRMION SOLUTIONS

The low energy effective lagrangian (3.14) gives rise to the following hamiltonian,

$$\mathcal{H} = \frac{\kappa\rho}{4}(\vec{\nabla}\hat{m})^2 - \frac{\mu_e}{2}\rho\cos\alpha\,\vec{B}\cdot\hat{m} + V(\rho_\uparrow,\rho_\downarrow) , \tag{4.1}$$

We recognize the hamiltonian of the usual sigma model, and conclude that – in the limit of vanishing Zeeman and Coulomb interactions – there are scale invariant skyrmion solutions. Just as in the fully polarized case, described by (1.1), the scale is set by a competition between Zeeman and Coulomb interactions.

According to Eq. 3.13, the deviations in both the charge density, $\delta\rho$, and in the spin density in the $\hat{m}$ direction, $\delta(\vec{S}\cdot\hat{m})$, are proportional to the Pontryagin density, q. Let us first discuss the consequences of the charge relation: A skyrmion with topological charge Q_{top} has electric charge Q_{el},

$$Q_{el} = \nu\cos\bar{\alpha}\,Q_{top} , \tag{4.2}$$

so for fully polarized states, this relation is the same as found by Sondhi $et\ al.$, but in general the charge is proportional to the polarization, $\bar{P} = \cos\bar{\alpha}$. In our case there is also another conserved quantity, namely the total spin in the $\hat{m}$ direction corresponding to the integral of the second relation in (3.13), $\int d^2x\,\delta(\vec{S}\cdot\hat{m}) = l_{22}Q_{top}$. That this quantity is quantized (and in general small) does not imply that the spin is small since $\hat{m}$ varies, and the spin is given by the Noether charge (2.6) which generates the global SU(2) symmetry as discussed in section 2. Also note that, since $\vec{S} \sim P\hat{m}$, the ratio of charge to spin depends only on the skyrmion profile, not on P.

V. MICROSCOPIC CONSIDERATIONS

In this section we shall give a heuristic derivation of (2.1) for the special case of a local potential and vanishing Zeeman coupling. The basic idea is to rewrite the functional integral using auxiliary fields in such a way that the expected ferromagnetic Goldstone mode is explicitly exhibited. Only when the theory is formulated in such a manner can we expect to correctly capture the long wave length physics in a mean field approximation. In the case of spin excitations in the Hubbard model, this was emphasized by Shultz [18], and in the present context by Moon $et\ al.$ [11]. In those papers it was also stressed that in order to reproduce Hartree-Fock results, one must use a particular form for the auxiliary field action. We shall see that also in our case the detailed result depends on the particular mean field decomposition. However, we want to stress from the outset that the general form (2.1) of the lagrangian is a generic result.

The initial steps are the same as in the derivation of the effective sigma model (1.1) for a single, filled and fully polarized Landau level as given by Moon $et\ al.$ [11]. In particular, we use a local repulsive potential, which is SU(2) invariant, and can be re-expressed in terms of spin variables:

$$V = \frac{V_0}{2}\rho(x)^2 = V_0\,\rho_\uparrow\rho_\downarrow(x)$$
$$= -\frac{V_0}{2}\,\vec{s}\cdot\vec{s}(x) + \frac{V_0}{8}\rho(x)^2 \ . \qquad (5.1)$$

Here, $\vec{s}$ is the spin operator $\vec{s} = \frac{1}{2}\psi^\dagger\vec{\sigma}\psi$, and $\rho = \psi^\dagger\psi$, where ψ is a two-component fermion field.[5] This particular decomposition differs from the one used by Moon *et al.*, and we will comment on this in section 6. Introducing the Hubbard-Stratonovich fields $\vec{h}$ and χ, the euclidean partition function can be written as,

$$Z[A^\mu, T] = \int \mathcal{D}\vec{h}\mathcal{D}\chi e^{-\int_0^{1/T} d\tau \int d^2x\,(\frac{1}{2V_0}h^2 + \frac{2}{V_0}\chi^2)}$$

$$\times \mathrm{Tr}\left\{ e^{-\frac{1}{T}\hat{H}_0}\mathrm{T}_\tau e^{-\int_0^{1/T} d\tau \int d^2x\,[-\vec{h}\cdot\vec{s}-(i\chi-\mu)\psi^\dagger\psi]} \right\} \ , \quad (5.2)$$

where the (2nd quantized) hamiltonian operator $\hat{H}_0$ is given by,

$$\hat{H}_0 = \int d^2x\,\psi^\dagger\frac{1}{2m}\left[\vec{p}-\vec{A}(\vec{x})\right]^2\psi(\vec{x}) \ . \qquad (5.3)$$

Here, T is the temperature and μ is the chemical potential. The trace in (5.2) is taken over all anti-periodic solutions to the following many-body problem: N non-interacting spin half fermions moving in two dimensions in an external (2-dimensional) gauge potential $\vec{A}(\vec{x})$ where the spins are coupled to an external (3-dimensional) magnetic field $\vec{h} = h\hat{m}$. Note that the trace is taken over all N corresponding to a grand canonical ensemble.

The external field problem defined by the trace in (5.2) is in general very hard to solve. To proceed we make the simplifying assumption that the spin of the particles in the $\hat{m}$-direction is a good quantum number which can be used to label the particles. This "adiabatic" assumption means that we neglect the spin-flip transitions induced by the Zeeman-like interaction $\vec{h}\cdot\vec{s}$,[6] and below we argue that the dynamics of our system is such that this is a good approximation for the low-energy sector. To evaluate the trace in (5.2) we use a first quantized path integral, where the effect of the spins of the particles can approximately be taken into account by including the appropriate Berry phases and Zeeman energies,

$$\mathrm{Tr}\left\{ e^{-\frac{1}{T}\hat{H}_0}\mathrm{T}_\tau e^{-\int_0^{1/T} d\tau \int d^2x\,[-\vec{h}\cdot\vec{s}-(i\chi-\mu)\psi^\dagger\psi]} \right\}$$

$$= \sum_{N=0}^\infty \int_{\Gamma_i} \prod_{i=1}^N \mathcal{D}\vec{x}_i(\tau) e^{-S_E[\vec{x}_i]} \ . \quad (5.4)$$

Here the sum over classical paths is understood to include sign factors appropriate to Fermi statistics, and the euclidean action is given by

$$S_E = \sum_{i=1}^N \left[\int_0^{1/T} d\tau\,(\frac{m}{2}\dot{\vec{x}}_i^{\,2} - i\vec{A}(\vec{x}_i)\cdot\dot{\vec{x}}_i + \mu \right.$$
$$\left. - i\chi(\vec{x}_i) + E_M(\vec{x}_i)) - i\gamma[\Gamma_i] \right] \ , \qquad (5.5)$$

where $\gamma[\Gamma_i]$ is the Berry phase picked up by particle i when it moves in the field $\vec{h}$ along the path Γ_i, and $E_M(\vec{x}_i)$ is the corresponding magnetization energy. Now comes the main observation: Using the parametrization (3.3), with the identification $\vec{h} = h\hat{m}$, the Berry phase $\gamma[\Gamma_i]$ can be written in the following way[7]

$$\gamma[\Gamma_i] = \mp \oint d\beta\,\sin^2\frac{k(\beta(\tau))}{2}$$

$$= \mp \int_0^{1/T} d\tau\,\partial_\tau\beta\sin^2\frac{k(\beta(\tau))}{2}$$

$$\mp \int_0^{1/T} d\tau\,\partial_\tau\vec{x}_i\cdot(\vec{\nabla}_{\vec{x}_i}\beta)\sin^2\frac{k(\beta(\tau))}{2}$$

$$= \pm \int_0^{1/T} d\tau\,\left(\tilde{a}^0(\tau,\vec{x}_i) + \frac{\partial\vec{x}_i}{\partial\tau}\cdot\vec{\tilde{a}}(\tau,\vec{x}_i)\right) \ , \quad (5.6)$$

where the vector potential $\tilde{a}^\mu$ is given by (3.6) and the sign depends on whether the spin of the particle is parallel or antiparallel to $\vec{h}$. Note that the angle $\beta(\tau,\vec{x}_i(\tau))$ has both a direct τ dependence from the time-dependence of $\hat{m}$, and an indirect one from the time-dependence of the particle position $\vec{x}_i(\tau)$.

In our particular system where the dynamics fixes both the density and the magnitude of the polarization (the corresponding modes have large gap as seen from (3.11)) we can calculate the magnetization energy $E_M(\tau)$ in the following approximation,

$$E_M(\vec{x}) = \mp\frac{1}{2}\left[h(\vec{x}) + \frac{1}{2}\ell^2\hat{m}(\vec{x})\cdot\vec{\nabla}^2\vec{h}(\vec{x}) + \dots \right] \ , \quad (5.7)$$

where plus and minus refer to the contributions from particles with spin in the direction of, or opposite to, the field $\vec{h}$ respectively. The first term on the right hand side is unambiguous, it is just the Zeeman energy, $\mp\frac{1}{2}h$, of the particle. The second term which is $\sim (\vec{\nabla}\vec{S})^2$, requires a more careful treatment involving a discussion of a short distance effect that goes beyond the arguments

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[5] The first equality in (5.1) holds only in a functional formulation, where ψ and $\psi^\dagger$ are Grassman numbers. If they are fermion operators, there are contact terms linear in the density that can be absorbed in the chemical potential.

[6] Note that we here neglect the effect of the real Zeeman coupling to the real external magnetic field $\vec{B} = \nabla\times\vec{A}$.

[7] The Berry phase entering in (5.5) is nothing but the noncovariant form of the spinfactor, as discussed in *e.g.* ref. [19]. In this reference it is also shown how to express the spinfactor in terms of a Grassmann integral. It would be interesting to formulate the present problem in this language and retain (5.5), after integrating over the appropriate Grassmann variables.

used so far: Because of the strong magnetic field, the electrons will have a rapid cyclotron motion, in addition to the slow motion of the guiding center. We assume that the spin, in the adiabatic approximation, will be aligned along the magnetic field averaged over this rapid motion. (This corresponds to a projection on the lowest Landau level, as done explicitly in ref. 11.) We can model this effect by smearing the electron over a distance given by the magnetic length, ℓ, and evaluate the Zeeman energy by assuming that the spin points in the direction of, or opposite to, the average $\vec{h}$,

$$E_M^\pm(\vec{x}) = \mp \frac{1}{2} \int d^2\delta\, f(\delta)\hat{m}(\vec{x}) \cdot \vec{h}(\vec{x}+\vec{\delta}) \qquad . \qquad (5.8)$$

Taking a gaussian profile, $f(\vec{\delta}) = \frac{1}{2\pi\ell^2} e^{-\delta^2/2\ell^2}$, for the smeared electron charge, Taylor expanding $\vec{h}(\vec{x}+\vec{\delta})$ and performing the δ integration, we get (5.7).

Equations (5.6) and (5.7) allow us to incorporate the effects of spin in the path integral in a very simple way: The Berry phase is included by coupling the particles to the gauge potential $\tilde{a}^\mu$, and $E_M(\vec{x})$ is taken outside the path integral in (5.4) since it depends only on $\vec{h}$.

To proceed, we have to distinguish between partial and full polarization, and we treat the two cases separately.

A. Partial polarization

We now return to a second quantized description and write the trace in (5.4) as a coherent state path integral. At this point we also switch to a bosonic description in terms of a two-component boson field, ϕ, and two auxiliary CS-fields a_α^μ. After the standard manipulations [20] we get the following expression for the partition function,

$$Z[A^\mu, T] = \int \mathcal{D}\vec{h}\mathcal{D}\chi e^{-\int_0^{1/T} d\tau \int d^2x\, (\frac{1}{2V_0}h^2 + \frac{2}{V_0}\chi^2)}$$
$$\times \int \mathcal{D}\phi\mathcal{D}\phi^\dagger \mathcal{D}a_\alpha^\mu e^{-\int d^3x\, \mathcal{L}[\phi,\nabla^\mu\phi,a_\alpha^\mu;A_\mu]}$$
$$\mathcal{L} = \mathcal{L}_0(\phi,\nabla^\mu\phi,a_\alpha^\mu;A_\mu) + E_M - (i\chi-\mu)\rho \quad (5.9)$$

where

$$\mathcal{L}_0 = \phi^\dagger i\nabla_0\phi - \frac{1}{2m_e}|\vec{\nabla}\phi|^2 - \frac{1}{2\pi} l_{\alpha\beta}\epsilon_{\mu\nu\sigma}a_\alpha^\mu \partial^\nu a_\beta^\sigma \quad (5.10)$$

$$i\nabla^\mu = i\partial^\mu + a_1^\mu + (a_2^\mu + \tilde{a}^\mu)\sigma_z + A^\mu \quad (5.11)$$

and

$$E_M = -\frac{1}{2}\rho\cos\alpha\left[h + \frac{1}{2}\ell^2\hat{m}\cdot\vec{\nabla}^2\vec{h} + \ldots\right] \quad . \quad (5.12)$$

The $\pm$ sign in (5.6) is here represented by $\phi^\dagger\sigma_z\phi = \rho\cos\alpha$. This introduces a dependence on the fixed z-direction (originating from the parametrization (3.3) of

$\hat{m}$), and one might think that the global SU(2) spin symmetry is explicitly broken. That this is not the case is seen by changing variables to the field $\tilde{\phi}$,

$$\tilde{\phi} = U(\vec{k})\phi = e^{\frac{i}{2}\vec{k}\cdot\vec{\sigma}}\phi \qquad , \qquad (5.13)$$

which does not change the integration measure, and noting that the covariant derivative ∇^μ transforms like,

$$U(\vec{k})\nabla^\mu U^\dagger(\vec{k}) = D^\mu \qquad , \qquad (5.14)$$

where D^μ is the covariant derivative given by (2.2), which depends only on the vector $\hat{m}$.

Finally, substituting (5.7) and (5.13) in (5.9), carrying out the gaussian integrals over the auxiliary fields χ and h, and dropping the tilde on $\tilde{\phi}$, we obtain our model (2.1) in the limit of vanishing Zeeman coupling, and with the potential,

$$V_{eff} = \frac{V_0}{8}[\rho^2 - (\rho\cos\alpha)^2]$$
$$= \frac{V_0}{8}\rho^2 - \frac{V_0}{2}\vec{S}\cdot\vec{S} = \frac{V_0}{2}\rho_\uparrow\rho_\downarrow \qquad , \qquad (5.15)$$

where $\vec{S}$ is the spin density in (2.6). Note that the net effect of the above manipulations is to replace the fermionic form of the spin density operator in (5.1) with the corresponding bosonic one in (5.15). Also note that (5.15) is manifestely invariant under the gauge and SU(2) transformations discussed in section 2. The coefficient κ is given by

$$\kappa = \frac{V_0}{2}\rho\ell^2\cos^2\alpha \qquad , \qquad (5.16)$$

which for $\cos\alpha = 1$ agrees with the spin-stiffness $\rho_s = \kappa/(4\pi\ell^2)$ calculated by Moon et al. for the fully polarized $\nu = 1$ state.

The Zeeman term can be added using the Noether construction as in section 2, or one can notice that to leading order $\langle\vec{s}\rangle = \frac{1}{2}\bar{\rho}\cos\bar{\alpha}\,\hat{m}$ so $\mu_e\vec{B}\cdot\vec{S} \simeq \frac{1}{2}\bar{\rho}\cos\bar{\alpha}\,\mu_e\hat{m}\cdot\vec{B}$, which is just the mean field value of the Zeeman term in (2.1).

B. Full polarization

In the case of fully polarized states, the previous derivation can be considerably simplified. We again switch to a second quantized description, but since all particles have spin along the direction of $\vec{h}(\vec{x},\tau)$ we can describe the system with a single scalar field coupled to a single Chern-Simons gauge potential choosen to change the statistics from fermions to bosons. The lagrangian corresponding to (5.11) becomes

$$\mathcal{L}_0 = \phi^\dagger i\nabla_0\phi - \frac{1}{2m_e}|\vec{\nabla}\phi|^2 - \frac{l}{2\pi}\epsilon_{\mu\nu\sigma}a^\mu\partial^\nu a^\sigma \quad (5.17)$$

$$i\nabla^\mu = i\partial^\mu + a^\mu + \tilde{a}^\mu + A^\mu \qquad , \qquad (5.18)$$

where ϕ is a single component field and l^{-1} an odd integer. Parametrizing ϕ as $\phi = \sqrt{\rho}e^{i\vartheta}$, and performing the same steps as in section 3, *i.e.* fixing unitary gauge, we arrive at,

$$
\mathcal{L} = \rho(a^0 + \tilde{a}^0) - \frac{1}{8m_e}(\vec{\nabla}\rho)^2 - V(\rho) - \frac{\rho}{2m_e}(\vec{a} + \vec{A} + \vec{\tilde{a}})^2
$$
$$
- \frac{l}{2\pi}\epsilon_{\mu\nu\sigma}a^\mu\partial^\nu a^\sigma - \frac{V_0}{2}(\partial_i\vec{S})^2 + \mu_e\vec{B}\cdot\vec{S} \quad . \qquad (5.19)
$$

This lagrangian corresponds to (3.7) for the case of partial polarization. A similar treatment of the high frequency mode as done for partial polarization in section 3 gives the same effective lagrangian (3.14), with $\alpha = 0$. The lagrangian (5.19) may in fact be seen as a special case of (3.7) with parameters $l_{11} = l_{22} = \pm l_{12}$. This is a singular case where the l matrix is not invertible. One of the Chern-Simons fields act as a multiplier to enforce the constraint of full polarization, and the ϕ field can be reduced to a single component field coupled to the second Chern-Simons field.

For a strictly local potential, the resulting potential as calculated from (5.15) is zero. This just reflects that fermions in a symmetric spin state do not interact via a local potential. In realistic cases the potential is of course not strictly local and this will introduce an effective potential in (5.19).

VI. CONCLUDING REMARKS

We have in this paper presented a mean field model for partially polarized, as well as fully polarized, quantum Hall states which seems in several respects preferable to previously used mean field models. It gives the correct low energy limit, described by a non-linear σ model with a spin wave spectrum determined by the Zeeman splitting. In this limit it is similar to the low-energy model previously used for the fully polarized case. It is of interest to stress some points concerning the comparison with other mean field models:

For the partially polarized states, there exist already a model based on a two-component bosonic field ϕ which is coupled to two Chern-Simons fields [14]. However, in this model there are no gapless spin waves (for vanishing Zeeman coupling.) In the present case, which also has a two-component ϕ field and two Chern-Simons fields, such low energy spin waves exist, due to the presence of the additional variable $\hat{m}$. The skyrmion solutions are also different in the two models. In the previous model the skyrmions are small, whereas in the present model the size of the skyrmions is determined by competition between the Coulomb and Zeeman terms, and may therefore be large (*i.e.* carry large spin).

For the fully polarized case, there exist a Landau-Ginzburg model based on a doublet scalar ϕ and a single

Chern-Simons gauge field [17]. This model has gapless spin waves, but the spin stiffness depends on the electron mass and not on the potential energy. That this scale is wrong is a serious problem that has been pointed out earlier (for example, although Sondhi *et al.* use the single CS field model to motivate the σ-model lagrangian, they use the phenomenological spin-stiffness in their original skyrmion calculations). For this reason we believe that our approach is to prefer also for the fully polarized states.

In addition to the Kohn mode (3.11) in the charge density, there is also a high frequency spin density wave. While the gap of the Kohn mode follows from a general sum rule argument, this is not the case for the spin density wave. However, both the high frequency modes are related to correlations in the ground state wave function, as stressed by Isakov [21], and the generalized Halperin (m_1, m_2, n) states can be obtained by considering gaussian fluctuations around the ground state mean field solutions for constant $\hat{m}$. This makes it plausible that the model correctly captures also the short distance part of the physics, even though one needs to go beyond the simple mean field approximation.

The microscopic derivation presented in section 5 gives a rationale for introducing the new field variable $\hat{m}$. Although the derivation is not rigorous, in our opinion it captures important elements of the correct physics. It is also closely related to the more detailed derivation of the low-energy effective lagrangian given by Moon *et al.* for the fully polarized case. At the level of details one should be aware of the following point: Even if the form of the spin stiffness term (5.16) follows from our general arguments, the numerical coefficient depends on the precise smearing of the magnetic field due to the cyclotron motion, as well as on the decomposition of the interaction (5.1) into a spin dependent and a spin independent part. There is no obvious choice for this decomposition in our derivation. We have chosen one which gives a bosonic effective potential $\sim \rho_\uparrow\rho_\downarrow$. With this choice, and using the probability density of a wave packet which is maximally localized in the lowest Landau level as smearing profile, the spin stifness comes out with the same numerical factor as in the work by Moon *et al.*.

We conclude by two comments on possible extensions of the present work.

1. The discussion in this paper has been entirely in the context of bosonic mean field theories. An alternative, and very successful, approach is the formulation in terms of composite fermions [24]. The CS-mean field description of spin polarized composite fermions was given by Lopez and Fradkin [25] and the generalization to partially polarized and unpolarized states by Mandal and Ravishankar [26]. Lopez and Fradkin also studied the closely related 2-layer problem [27], and explicit composite fermion wave functions describing spin waves were studied numerically by Nakajima and Aoki [28]. The

formalism developed in this paper can immediately be carried over to the fermionic CS-theory, and would give an alternative description of spin waves in the mean field theory of composite fermions. A possible advantage of such a formulation is that it would be manifestly SU(2) invariant from the outset, and it might be of interest to compare such a formulation to the ones by Mandal and Ravishankar, and by Ray [13].

2. One interesting application of the various effective field theories for the QH system is to study edge modes. For sharp edges a dual CS description is very useful, and leads to a description of the edge modes in terms of chiral bosons. A dual CS description incorporating spin was given by Stone, [29] and recently used by Milovanović [30] to derive the corresponding edge theory. It might be of interest to derive a dual CS theory and the corresponding edge theory from the CS theory given in this paper. Recently, it has been argued that, as the confining potential softens, a sharp polarized edge reconstructs by developing a spin texture [31,32]. This phenomenon has been studied in the bosonic CS theory by Leinaas and Viefers [33]. Using the CS theory proposed in this paper, that analysis could be extended to partially polarized states, and even in the fully polarized case, using the lagrangian (5.19) would have the advantage of having the correct spin stiffness.

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